

Available online at www.sciencedirect.com

SCIENCE @ DIRECT®

Topology 45 (2006) 261–280

TOPOLOGY

www.elsevier.com/locate/top

Estimates of characteristic numbers of real algebraic varieties

Yves Laszlo, Claude Viterbo*

Centre de Mathématiques Laurent Schwartz, UMR 7640 du CNRS, Ecole Polytechnique - 91128 Palaiseau, France

Received 14 December 2004; received in revised form 5 August 2005

Abstract

We give some explicit bounds for the number of cobordism classes of real algebraic manifolds of real degree less than d , and for the size of the sum of Betti numbers with $\mathbb{Z}/2$ coefficients for the real form of complex manifolds of complex degree less than d .

© 2005 Elsevier Ltd. All rights reserved.

Keywords: Real algebraic manifolds; Pontriagin numbers; Complexity

0. Introduction

In complex as well as in real algebraic geometry, it is useful to understand what are the simplest manifolds, and try to measure the complexity of algebraic manifolds. A priori there are many ways to define what “simple” manifolds should be and for no obvious reason should they all agree. Just to mention a few examples, among the most frequently used (see [18]), there are low degree manifolds, rational manifolds, Fano and uniruled manifolds. These manifolds have sometimes bounded Betti numbers, and some belong to finitely many deformation classes (low degree and Fano).

Once we decide on a given notion of complexity for complex manifolds, we may try to understand the properties of their real part. Are the real parts of “simple” complex manifolds “simpler”?

* Corresponding author.

E-mail addresses: laszlo@math.polytechnique.fr (Y. Laszlo), viterbo@math.polytechnique.fr (C. Viterbo).

The situation is far from being understood. Among the oldest results in real algebraic geometry, we find the problem of estimating the Betti numbers of a real algebraic manifold.

Besides Harnack's result for the maximal number of connected components of a real algebraic curve, dating back to 1876, and later work, for example by Severi and Comessatti for surfaces, the literature we are aware of, may be divided into two classes.

In the first approach, the n -dimensional manifold $M_{\mathbb{R}}^n$ is the set of real points of a complex manifold, $M_{\mathbb{C}}^n$. The inequality between the Betti numbers with $\mathbb{Z}/2$ coefficients

$$\sum_{j=0}^n b_j(M_{\mathbb{R}}^n) \leq \sum_{j=0}^{2n} b_j(M_{\mathbb{C}}^n) \quad (1)$$

is then given by the so-called Smith's theory. The first occurrence of this theory can be found in [26,24] explicitly for the number of connected components of a real algebraic set, but the same ideas can be applied to estimate higher Betti numbers, as hinted in the paper (see particularly [24, p. 620; 26, p. 509]). The explicit inequality (1) can first be found in [11, Theorem 4.4, p. 146] and the modern presentation, using spectral sequences for the $\mathbb{Z}/2$ -equivariant cohomology, is due to [2, p. 55, Section 4.1]. These results imply that whenever there are bounds on the Betti numbers with $\mathbb{Z}/2$ coefficients of the set of complex points, there are bounds on the set of real points.

Another approach defines $M_{\mathbb{R}}^n$ by equations $F_1 = 0, \dots, F_k = 0$ of degree d_1, \dots, d_k , in $\mathbb{R}P^{n+k}$. The Oleinik inequalities for hypersurfaces, extended to the general case by Thom and Milnor (see [22,21,29,19]), are all based on applying Morse theory, usually to $F = \sum_{j=1}^k F_j^2$. Thus, their estimate will depend on $\delta = \max d_k$ and yield a bound of the order of δ^{n+k-1} .

Definition 0.1. The *real degree* of a smooth real algebraic manifold M is the minimal degree of a projective embedding of M in a real projective space. By definition, this is also the minimal degree of a projective embedding of all algebraic manifolds $M_{\mathbb{C}}$ such that $M_{\mathbb{R}} = M$ in the complex projective space, which are equivariant for the anti-holomorphic involution of $M_{\mathbb{C}}$ (i.e. such that $\text{Fix}(\tau) = M$) and the canonical anti-holomorphic involution of the projective space. It is also equal to the smallest degree of a very ample line bundle L on $M_{\mathbb{C}}$ such that $\tau^*(L) = \bar{L}$.

Now, applying the above results and considering the fact that a general algebraic manifold of *real* degree d can be realized as a real algebraic manifold in $\mathbb{R}P^{2n+1}$, defined by equations of degree less than d , we get an estimate of the type¹

$$\sum_{j=0}^n b_j(M_{\mathbb{R}}^n) \leq C d^{2n+1}.$$

We will show in the second part of the paper that bounds on the *complex* degree of the complex manifold are sufficient to yield bounds on the Betti numbers with $\mathbb{Z}/2$ coefficients, and these bounds are explicit and of the order of $C d^{n+1}$.

It is also known according to Eliashberg and the second author (cf. [14,10,17] for the three-dimensional case) that complex uniruled manifolds cannot have a real form with negative sectional curvature and this

¹ Alternatively, if one wants to use the *complex* degree, one could use the Oleinik–Thom–Milnor inequality for the complex case, and then apply Smith's theory, but the estimates would then be in d^{4n+2} .

means that negatively curved manifolds are “complicated”: in particular, they cannot be represented by equations of low degree.

Besides this, little is known about the following conjectures, where the word *simple* has no precise meaning.²

Conjecture 1. *Let $M(\mathbb{C})$ be a smooth projective manifold. Then the set of diffeomorphism types of its real forms is finite.*

Conjecture 2. *Simple complex manifolds have simple real forms.*

To prove this even in the more accessible simply connected case, one would need bounds on the Betti numbers, which follow from the Smith–Thom inequalities, the multiplicative structure and the Pontrjagin classes. Our goal here is to prove that for manifolds of low degree, the Pontrjagin numbers are explicitly bounded by the degree of a real embedding, and thus the number of cobordism classes. If the complex cotangent bundle is nef, this bound depends only on the degree of a complex embedding: thus, the Pontrjagin numbers are also a measure of the complexity at least in this case.

1. Main theorem

In this paper we shall work with complex projective manifolds $M_{\mathbb{C}}$, with an anti-holomorphic involution τ . The set of fixed points is denoted by $M_{\mathbb{R}}$. Another point of view is to look at a smooth projective real variety X defining

$$M_{\mathbb{C}} = X(\mathbb{C}) \quad \text{and} \quad M_{\mathbb{R}} = X(\mathbb{R}).$$

We will be mainly interested in the case where $M_{\mathbb{R}}$ is further assumed to be orientable: we will say simply in this case that M (or $M_{\mathbb{R}}$) is an orientable real algebraic manifold.

We will use without any further comment these two points of view (with this notation). Of course for a given $M_{\mathbb{C}}$ there may be many non-equivalent $M_{\mathbb{R}}$; in fact as pointed out in the introduction, it is not even known that there are finite diffeomorphism, homeomorphism or homotopy types of $M_{\mathbb{R}}$. Vice-versa, it is not known, given $M_{\mathbb{R}}$, what are the different complex manifolds $M_{\mathbb{C}}$ having $M_{\mathbb{R}}$ as real form. But here even the formulation of the problem is unclear, since we may obviously blow up $M_{\mathbb{C}}$ along subvarieties invariant by τ but not intersecting $M_{\mathbb{R}}$, without changing $M_{\mathbb{R}}$. Even in the two-dimensional case, where we may ask how many minimal $M_{\mathbb{C}}$ yield the given real form³: the Klein bottle is the real form of all the odd Hirzebruch surfaces, $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(2k+1))$ [25].

Now given a real algebraic manifold, we shall assume it is endowed with a *real* very ample line bundle H , where *real* means that $\tau^*(H) = \overline{H}$. Then we denote $c_1(H)^n$ by $\deg(M)$. Note that if H is a (non-real) very ample line bundle over $M_{\mathbb{C}}$, then $L = H \otimes \tau^*(\overline{H})$ is a real very ample line bundle, but there is no general bound of its degree depending only on the degree of H .

Our first result is:

² However, we refer to [4–7] for some examples.

³ If one wants to have surfaces that are not deformation equivalent one could consider $\mathbb{P}^1 \times \mathbb{P}^1$ and the product of two elliptic curves with the standard anti-holomorphic involution.

Theorem 1.1. *Let $M_{\mathbb{R}}$ be an orientable real algebraic manifold of degree d and dimension n . Then the Pontriagin numbers satisfy the following inequalities:*

$$|p_I(M_{\mathbb{R}})| \leq 2^{n^2+3n} d(d+n-2)^n.$$

In particular, the number of cobordism classes of manifolds representable as real algebraic manifolds of a given real degree can be explicitly bounded.

Note that the existence of such a bound can be proved a priori, since one can prove that real manifolds of a given (real) degree fall into finitely many diffeomorphism classes, but we believe that making these bounds explicit is useful. The only result we know of, related to the above, is due to Thom, who proves in [28] that if V is a compact real affine complete intersection (i.e. given by k polynomial equations in \mathbb{R}^{n+k}), then it is a (unoriented) boundary, and thus all Stiefel–Whitney numbers are zero.

The next step in our program should then be to show that such a bound also holds for d the complex degree. This would then imply that the real part of a complex manifold of given degree falls in finitely many cobordism classes, a first step towards the proof of conjecture 1. For the moment, there are only a few classes for which this can be proved. Note that, as pointed out by Kharlamov, using Torelli’s theorem for K3 surfaces, it is easy to construct surfaces with complex degree bounded and the real degree can be arbitrarily large.

We also give some estimate on the size of the homology for the real part of a complex manifold. Note that contrary to the previous theorem, the degree used here is the complex degree, that is the minimal degree of any algebraic embedding, not necessarily real.

Let $b(X)$ represent the sum of the Betti numbers with $\mathbb{Z}/2$ coefficients.

Theorem 1.2. *Let X be a real subvariety of the projective space. Assume that $X_{\mathbb{C}}$ is smooth, connected to complex degree d and dimension n . Then, one has $b(X(\mathbb{R})) \leq b(X(\mathbb{C})) \leq 2^{n^2+2} d^{n+1}$.*

The first inequality is just the Smith–Thom inequality. As for the second one, according to Milnor [19], a complex submanifold in $\mathbb{C}P^m$ given by equations of degree less than δ has $b(X_{\mathbb{C}}) \leq C_n \delta^{2m+2}$. Since if $X(\mathbb{C})$ has degree d , it is given by polynomial equations of degree at most d in $\mathbb{C}P^{2n+1}$ (see [20]), Milnor’s inequality yields $b(X_{\mathbb{C}}) \leq C_n d^{4n+4}$. In the real case, for a submanifold in $\mathbb{R}P^m$, Milnor gives $b(X_{\mathbb{R}}) \leq C_n \delta^{m+1}$, and the same argument yields $b(X_{\mathbb{R}}) \leq C_n d^{2n+2}$. However, we cannot say that we improve on Milnor’s inequality, since there seems to be no clear lower bound for the maximal degree of the equations defining X in $\mathbb{C}P^{2n+1}$ as a function of the degree of X .

2. Some lemmata on real and complex cycles

Since we shall deal with non-orientable situations, we shall use homology and cohomology with local systems of coefficients,⁴ \mathcal{L} . Unless otherwise stated, all our manifolds are closed and connected. We denote by $H^*(X, \mathcal{L})$ and $H_*(X, \mathcal{L})$ such groups, by $H_c^*(X, \mathcal{L})$ its compact supported version. We denote by o_X the orientation bundle of X .

⁴ We shall only use the case of local coefficients with fiber \mathbb{Z} , and this implies that we always have $\mathcal{L} \otimes \mathcal{L} = \mathbb{Z}$. In particular we identify \mathcal{L} to \mathcal{L}^* . Of course, with $\mathbb{Z}/2\mathbb{Z}$ coefficients, \mathcal{L} is trivial and much of what follows is much simpler.

Moreover, we have cup and cap product map:

$$\begin{aligned}\cup : H^k(M, \mathcal{L}) \otimes H^l(M, \mathcal{L}') &\longrightarrow H^{k+l}(M, \mathcal{L} \otimes \mathcal{L}'), \\ \cap : H^k(M, \mathcal{L}) \otimes H_l(M, \mathcal{L}') &\longrightarrow H_{l-k}(M, \mathcal{L} \otimes \mathcal{L}').\end{aligned}$$

The first map is a non-degenerate pairing for $k + l = n$, $\mathcal{L} \otimes \mathcal{L}' = o_M$.

Also, a map $f : V \rightarrow M$ induces maps

$$\begin{aligned}f_* : H_*(V, f^*(\mathcal{L})) &\longrightarrow H_*(M, \mathcal{L}), \\ f^* : H^*(V, \mathcal{L}) &\longrightarrow H^*(M, f^*(\mathcal{L})).\end{aligned}$$

Moreover, we have a Poincaré duality isomorphism:

$$PD : H_k(M, \mathcal{L}) \longrightarrow H^{n-k}(M, \mathcal{L} \otimes o_M).$$

This implies in particular that $H_n(M, o_M) = H^0(M, \mathbb{Z})$ is canonically isomorphic to \mathbb{Z} .

Finally, to a k -dimensional vector bundle, $\pi : E \rightarrow X$, we associate the relative orientation bundle $o_{E/X}$, as the set of orientations of the fiber, namely $o_E \otimes \pi^*(o_X^{-1})$. The Thom isomorphism

$$T : H^*(X, \mathcal{L}) \longrightarrow H_c^{*+k}(E, \mathcal{L} \otimes o_{E/X})$$

is given by $T(u) = \pi^*(u) \cup T_E$, where $T_E \in H_c^k(E, o_{E/X})$. The class T_E is uniquely defined as the preimage of 1 by the canonical isomorphism $H_c^k(E, o_{E/X}) \rightarrow \mathbb{Z}$. Naturality implies that for every x in X , T_E is uniquely determined by the property that it restricts on $H_c^k(E_x, o_{E_x})$ to the canonical generator.

From this it easily follows,⁵ using the canonical isomorphism $\pi^*o_{E/X} \otimes \pi'^*(o_{E'/X}) \simeq o_{E \oplus E'/X}$, that $T_{E \oplus E'} = T_E \cup T_{E'}$, since we just have to check this over a point.

Finally note that Poincaré duality on E is an isomorphism

$$H_{n-d}(E, \mathcal{L}) \rightarrow H_c^{k+d}(E, \mathcal{L} \otimes o_M)$$

but since M is a retract of E , and

$$H_{n-d}(E, \mathcal{L}) \simeq H_{n-d}(M, \mathcal{L}) \simeq H^d(M, \mathcal{L} \otimes o_M)$$

the second isomorphism being again Poincaré duality, we get an isomorphism

$$H^d(M, \mathcal{L} \otimes o_M) \rightarrow H_c^{k+d}(E, \mathcal{L} \otimes o_M),$$

and so the Thom isomorphism is induced by Poincaré duality.

Now if \mathcal{L} is a local system and we have isomorphism $\varepsilon : H_n(M, \mathcal{L}) \rightarrow \mathbb{Z}$ that we call an \mathcal{L} pseudo-orientation, and a map $j : V \rightarrow M$ and $j^*(\mathcal{L}) \otimes o_V \simeq \mathbb{Z}$ (or equivalently $j^*(\mathcal{L}) \simeq o_V$), then we say that V is \mathcal{L} -coorientable. If moreover one of these two isomorphisms is given, we say that V is \mathcal{L} -cooriented. In the case $\mathcal{L} = o_M$ we just say that V is coorientable (resp. cooriented).

Now if V is k -dimensional, connected and \mathcal{L} -cooriented, we may associate with V a class $[V] \in H_k(M, \mathcal{L})$ as follows. By Poincaré duality we have an isomorphism

$$H_k(V, j^*(\mathcal{L})) \rightarrow H^0(V, j^*(\mathcal{L}) \otimes o_V) \simeq \mathbb{Z}$$

⁵ Note that if π, π' are the projections of $E \oplus E'$ on E, E' , then $\pi^*(T_E)$ has compact support only in the E direction, and $(\pi')^*(T_{E'})$ in the E' direction, but then $\pi^*(T_E) \cup (\pi')^*(T_{E'})$ has compact support, and we denote this class by $T_E \cup T_{E'}$.

and the last isomorphism is canonically given by the \mathcal{L} -coorientation of V . In the non-connected case, we just do the same for each connected component.

Then we have $j_* : H_k(V, j^*(\mathcal{L})) \rightarrow H_k(M, \mathcal{L})$ and we still denote by $[V]$ the class $j_*([V]) \in H_k(M, \mathcal{L})$.

The submanifold V also has a fundamental class μ_V in $H^{n-k}(M, \mathcal{L} \otimes o_M)$ defined as the extension of the Thom class of the normal bundle of V in M . It is the image of $[V] \in H_k(M, \mathcal{L})$ by Poincaré duality. fundamental class $[V] \in \mathbb{Z}$.

Note that if $\beta \in H^k(M, \mathcal{L}^*)$ then $\langle \beta, [V] \rangle \in H_0(M, \mathbb{Z}) \simeq \mathbb{Z}$ is given by $\mu_V \cup \beta \in H^n(M, \mathcal{L} \otimes \mathcal{L}^* \otimes o_M) \simeq H^n(M, o_M) \simeq \mathbb{Z}$.

Lemma 2.1. *Let V and W be \mathcal{L} -cooriented and \mathcal{L}' -cooriented submanifolds of M . Assume V and W intersect transversally, and set $Z = V \cap W$. Then Z is $\mathcal{L} \otimes \mathcal{L}' \otimes o_M$ cooriented and if $\mu_V \in H^k(M, \mathcal{L} \otimes o_M)$ and $\mu_W \in H^l(M, \mathcal{L}' \otimes o_M)$, then $\mu_V \cup \mu_W = \mu_Z \in H^{k+l}(M, \mathcal{L} \otimes \mathcal{L}')$.*

Proof. The class μ_V is related to the Thom class as follows: if V is an \mathcal{L} -cooriented submanifold of M , $N(V, M)$ its normal bundle and $T_{(V, M)} \in H_c^{n-k}(N(V, M), \otimes_{N(V, M)})$ its Thom class, then the extension of T_V to $H^{n-k}(M, \otimes_{o_M})$ is precisely μ_V .

Moreover, by naturality of the Thom class, if $i : Z \rightarrow V$, E is a bundle over V , and we denote by \tilde{i} the natural map $i^*(E) \rightarrow E$, then $T_{i^*(E)} = (\tilde{i})^*(T_E)$.

Now let $j_V : V \cap W \rightarrow V$, $j_W : V \cap W \rightarrow W$ be the inclusions, since $j_V^*(N(V, M)) = N(V \cap W, W)$, $j_W^*(N(W, M)) = N(V \cap W, V)$, and of course by general position

$$N(V \cap W, M) = N(V \cap W, W) \oplus N(V \cap W, V),$$

that is

$$N(V \cap W, M) = j_W^* N(W, M) \oplus j_V^* (N(V, M))$$

and this induces an isomorphism $o_{N(V \cap W, M)/V \cap W} \simeq j_V^* o_{N(V, M)/V \cap W} \otimes j_W^* o_{N(W, M)/V \cap W}$ and we get that $T_{N(V \cap W, M)} = T_{N(W, M)} \cup T_{N(V, M)}$, that is

$$T_{V \cap W} = T_V \cup T_W.$$

Thus, if V , W are \mathcal{L} and \mathcal{L}' -coorientable submanifolds, in general position, we have

$$T_{V \cap W} = T_V \cup T_W.$$

This implies that

$$\mu_{V \cap W} = \mu_V \cup \mu_W. \quad \square$$

In particular, if V is oriented (and therefore \mathbb{Z} -cooriented) and W cooriented, we have that $\mu_V \in H^*(M, o_M)$, $\mu_W \in H^*(M, \mathbb{Z})$. Then Z is oriented and dual to $\mu_V \cup \mu_W \in H^*(M, o_M)$.

Remark 2.2. This extends to the case where V and W are singular submanifolds with codimension 2 singular locus. Indeed in this case the fundamental class is also well defined, and provided V and W are in general position, $V \cap W$ is also a singular manifold with codimension 2 singular locus and the above result also applies. (See [3] for more details).

Let us now consider a smooth real projective manifold M of dimension n .

Let V and W be two real submanifolds of M , so that we have $V_{\mathbb{C}}, W_{\mathbb{C}} \subset M_{\mathbb{C}}$ and $V_{\mathbb{R}}, W_{\mathbb{R}} \subset M_{\mathbb{R}}$. Note that $V_{\mathbb{C}}, W_{\mathbb{C}}$ and $M_{\mathbb{C}}$ are always orientable.

Let us denote by “ \cdot ” the algebraic intersection of cycles, and assume $V_{\mathbb{C}}$ and $W_{\mathbb{C}}$ are in general position and have complementary dimension. For example, if $V_{\mathbb{C}}$ and $W_{\mathbb{C}}$ are zero sets of sections of very ample bundles, we can always move V and W so that this assumption is satisfied.

Lemma 2.3. *Assume $V_{\mathbb{R}}$ is \mathcal{L} -coorientable, $W_{\mathbb{R}}$ is \mathcal{L}^* -orientable and $V_{\mathbb{R}}$ and $W_{\mathbb{R}}$ intersect transversally. Then*

$$|V_{\mathbb{R}} \cdot W_{\mathbb{R}}| \leq V_{\mathbb{C}} \cdot W_{\mathbb{C}}.$$

Proof. The assumption is only needed so that the number $V_{\mathbb{R}} \cdot W_{\mathbb{R}}$ is well defined. Indeed assume that the submanifolds are in general position. Then $|V_{\mathbb{R}} \cdot W_{\mathbb{R}}|$ counts with sign the number of intersection points of $V_{\mathbb{R}}$ and $W_{\mathbb{R}}$. But these are contained in the set of intersection points of $V_{\mathbb{C}}$ and $W_{\mathbb{C}}$, while $V_{\mathbb{C}} \cdot W_{\mathbb{C}}$ counts all intersection points with positive sign. The inequality is now obvious. \square

Remark 2.4. The same proof shows that if $V_{\mathbb{R}}^j$ are \mathcal{L}_j coorientable, with

$$\mathcal{L}_1 \otimes \mathcal{L}_2 \otimes \cdots \otimes \mathcal{L}_k \otimes (o_M)^k = \mathbb{Z},$$

then we have

$$|V_{\mathbb{R}}^1 \cdot V_{\mathbb{R}}^2 \cdots V_{\mathbb{R}}^k| \leq V_{\mathbb{C}}^1 \cdot V_{\mathbb{C}}^2 \cdots V_{\mathbb{C}}^k.$$

Let f be a “real” holomorphic map from a real manifold V to M ; that is $f : V_{\mathbb{C}} \rightarrow M_{\mathbb{C}}$ commutes with the anti-holomorphic involution, hence, sends $V_{\mathbb{R}}$ to $M_{\mathbb{R}}$. Then f sends $[V_{\mathbb{C}}]$ to a class $(f_{\mathbb{C}})_*([V_{\mathbb{C}}])$, and $[V_{\mathbb{R}}]$ to $(f_{\mathbb{R}})_*([V_{\mathbb{R}}])$. Consider a similar “real” holomorphic map g from W to M . We claim that the above result also implies

Proposition 2.5. *Assume $V_{\mathbb{R}}$ is $f_{\mathbb{R}}^*(\mathcal{L})$ coorientable, and $W_{\mathbb{R}}$ is $f_{\mathbb{R}}^*(\mathcal{L}^*)$ orientable. If $f(V_{\mathbb{C}}) \cap g(W_{\mathbb{C}})$ is finite, then we have*

$$|(f_{\mathbb{R}})_*([V_{\mathbb{R}}]) \cdot (g_{\mathbb{R}})_*([W_{\mathbb{R}}])| \leq (f_{\mathbb{C}})_*([V_{\mathbb{C}}]) \cdot (g_{\mathbb{C}})_*([W_{\mathbb{C}}]).$$

Proof. With the same argument as above, we just have to prove that the multiplicity of a complex intersection is always larger than the multiplicity of the real intersection. But this is a local statement, and we can then of course move locally V and W to put them in general position, in which case the statement is obvious. \square

Remark 2.6. If $V \cap W$ contains a component of positive dimension, the above inequality may fail, since $V_{\mathbb{C}} \cdot W_{\mathbb{C}}$ can be negative. One could however hope for an inequality with $V_{\mathbb{C}} \cdot W_{\mathbb{C}}$ replaced by $|V_{\mathbb{C}} \cdot W_{\mathbb{C}}|$. Note that the main example is given by a “real” holomorphic bundle E over $V_{\mathbb{C}}$ and $W = V$. Then $V \cdot W$ denoted by $\chi(E)$ is the number of zeros of a generic smooth section of E counted with sign. In the complex

case this coincides with $c_n(E)$ ($n = \dim(E)$). In both real and complex cases, if $E = TV$, it coincides with

$$\sum_{j=0}^{\dim(V)} (-1)^j \dim H^j(V, \mathbb{Z}).$$

Then $V_{\mathbb{C}} \cap W_{\mathbb{C}} = \chi(E_{\mathbb{C}})$ and $V_{\mathbb{R}} \cap W_{\mathbb{R}} = \chi(E_{\mathbb{R}})$. Thus we could expect

$$|\chi(E_{\mathbb{R}})| \leq |\chi(E_{\mathbb{C}})|.$$

However, there is the following counter-example, suggested by Slava Kharlamov: let $M_{\mathbb{C}}$ be a four-dimensional complex manifold such that $M_{\mathbb{R}}$ is orientable and $|\chi(M_{\mathbb{R}})|$ is arbitrarily large.

Let P be the blow-up of M along the curve C . The formula $\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B)$ follows from the Mayer–Vietoris exact sequence and implies that

$$\chi(P_{\mathbb{C}}) = \chi(M_{\mathbb{C}}) + (v - 1)\chi(C_{\mathbb{C}}),$$

where $v = \text{codim}(C) = 3$, and

$$\chi(P_{\mathbb{R}}) = \chi(M_{\mathbb{R}}) + 2\chi(C_{\mathbb{R}}).$$

Since $C_{\mathbb{R}}$ is a union of circles, $\chi(C_{\mathbb{R}}) = 0$ and thus $\chi(P_{\mathbb{R}}) = \chi(M_{\mathbb{R}})$, while by a suitable choice of C , we may arrange that $\chi(P_{\mathbb{C}})$ is small. Indeed, take $M_{\mathbb{C}}$ to be the product of a surface with a real part of arbitrarily large Euler characteristic, and the surface

$$S = \{(x, y, z, t) \in \mathbb{C}P^3 \mid x^2 + y^2 + z^2 + t^2 = 0\}.$$

Then $S_{\mathbb{R}}$ is a sphere, $\chi(S_{\mathbb{R}}) = 2$ and $S_{\mathbb{C}}$ contains a curve of genus 4; taking the intersection of $S_{\mathbb{C}}$ with a generic hypersurface of degree 3 yields a curve of genus 4, thus $\chi(C) = -6$. Since we may move C , after blowing up we may find a new curve of genus 4 in the blown-up manifold, and thus we may repeat the blow-up, and eventually get a manifold P such that $|\chi(P_{\mathbb{C}})|$ is between 0 and 5, while $|\chi(P_{\mathbb{R}})| = |\chi(M_{\mathbb{R}})|$ is large.

Thus, we see that $|\chi(P_{\mathbb{R}})| = |\chi(TP_{\mathbb{R}})|$ can be arbitrarily large, depending on the choice of the first surface, while $|\chi(P_{\mathbb{C}})| = |\chi(TP_{\mathbb{C}})|$ is between 0 and 5, and $TP_{\mathbb{R}}$ and $TP_{\mathbb{C}}$ are the counter-examples we were looking for.

Lemma 2.7. *Let X and Y be, respectively, \mathcal{L} -cooriented and \mathcal{L}' -cooriented submanifolds of M , and Z be the clean intersection of X and Y , that is $T_Z X \cap T_Z Y = TZ$. Then, if μ_X and μ_Y are the cohomology classes associated with X and Y in $H^*(M, \mathcal{L} \otimes o_M)$ and $H^*(M, \mathcal{L}' \otimes o_M)$, we have $\mu_X \cup \mu_Y = \mu_N \in H^*(M, \mathcal{L} \otimes \mathcal{L}')$, where N is the zero set of a generic section of νZ , where, $\nu Z = T_Z M / (T_Z X + T_Z Y)$.*

Proof. Let us take local coordinates in a neighborhood of Z , associated with the decomposition $T_Z M = TZ \oplus \nu_X \oplus \nu_Y \oplus \nu Z$, where ν_X (resp. ν_Y) is the normal bundle of Z in X (resp. Y). Then X is parameterized by the total space of $\nu_X : (z, \xi_X) \rightarrow (z, \xi_X, 0, 0)$ and Y by the total space of $\nu_Y : (z, \xi_Y) \rightarrow (z, 0, \xi_Y, 0)$. A generic perturbation \tilde{X} of X will be given by $(z, \xi_X) \rightarrow (z, \xi_X, 0, \varepsilon(z, \xi_X))$, where $\varepsilon(z, \xi_X)$ vanishes for $|\xi_X| \geq \delta$. Then the intersection $\tilde{X} \cap Y$ is given by the equation $\varepsilon(z, 0) = 0$, where ε is a section of νZ . This intersection is transverse provided $\frac{\partial \varepsilon}{\partial z}(z, 0)$ has maximal rank at points where $\varepsilon(z, 0) = 0$, and $\tilde{X} \cap Y$ is equal to the zero set of the section ε . We let the reader check that the coorientations match. \square

In the sequel, given a submanifold V of M , we denote by $V \cdot V$ the intersection of V with the image of V by a small generic isotopy. If V is \mathcal{L} coorientable, then $V \cdot V$ is $\mathcal{L}^{\otimes 2} \otimes o_M$ coorientable. The class μ_V is in $H^*(M, \mathcal{L} \otimes o_M)$, and $V \cdot V$ is Poincaré dual to $\mu_V^2 \in H^*(M, \mathcal{L}^{\otimes 2})$.

We now denote by c_V and c_W the codimension of $V_{\mathbb{R}}$ and $W_{\mathbb{R}}$ in $M_{\mathbb{R}}$, so that these are also the complex codimensions of $V_{\mathbb{C}}$ and $W_{\mathbb{C}}$ in $M_{\mathbb{C}}$. Let $\gamma_V \in H^{2c_V}(M_{\mathbb{C}}, \mathbb{Z})$ be the class Poincaré dual to $V_{\mathbb{C}}$. Assume that V is coorientable and let $\rho_V \in H^{c_V}(M_{\mathbb{R}}, o_V)$ be the class Poincaré dual to $V_{\mathbb{R}}$. Then $[V_{\mathbb{R}}] \cdot [V_{\mathbb{R}}] \in$ defined as $(T_{V_{\mathbb{R}}} \cap [V_{\mathbb{R}}])$ is dual to $\rho_V^2 \in H^{2c_V}(M_{\mathbb{R}}, \mathbb{Z})$.

Let i be the inclusion of $M_{\mathbb{R}}$ into $M_{\mathbb{C}}$; we want to compute $i^*(\gamma_V)$ in terms of ρ_V .

Proposition 2.8. *We have*

$$i^*(\gamma_V) = \rho_V^2.$$

Proof. Applying Lemma 2.7 to $X = V_{\mathbb{C}}$ and $Y = M_{\mathbb{R}}$, with $Z = V_{\mathbb{R}}$, we have that if $\nu(V_{\mathbb{R}})$ is the normal bundle of $V_{\mathbb{R}}$ in $M_{\mathbb{R}}$, it is easy to see that $J\nu(V_{\mathbb{R}})$ is normal to the space $T(V_{\mathbb{C}}) + T(M_{\mathbb{R}})$ in $TM_{\mathbb{C}}$, and, thus, a perturbation of $V_{\mathbb{R}}$ will intersect $V_{\mathbb{C}}$ along the zeros of a section of the bundle $J\nu(V_{\mathbb{R}})$ over $V_{\mathbb{R}}$. Since this bundle is isomorphic to $\nu(V_{\mathbb{R}})$, this is the same as the zero set of a section of the normal bundle, and this coincides with $V_{\mathbb{R}} \cdot V_{\mathbb{R}}$. \square

Proposition 2.9. *If $V_{\mathbb{R}}$ is \mathcal{L} coorientable, then we have*

$$(f_{\mathbb{C}})_*([V_{\mathbb{C}}]) \cap M_{\mathbb{R}} = (f_{\mathbb{R}})_*([V_{\mathbb{R}}]) \cdot (f_{\mathbb{R}})_*([V_{\mathbb{R}}]).$$

Remark 2.10. Note that this is still true for $V_{\mathbb{C}}$, a variety with singularities of codimension at least 2, so that the same holds for $V_{\mathbb{R}}$. This does not hold if the singularities have codimension 1: in this case we cannot even guarantee that $V_{\mathbb{R}}$ is a cycle.

Remark 2.11. Given the map $f : X \rightarrow Y$ the normal bundle is the quotient $f^*(TY)/TX$.

Proof of Proposition 2.9. If f is an embedding, this follows from the previous proposition. Consider now the case where $V \subset E$ and $f = \pi|_V$, where $\pi : E \rightarrow B$ is a real holomorphic projection (so that $\pi^{-1}(B_{\mathbb{R}}) = E_{\mathbb{R}}$). Let $Z \subset B_{\mathbb{R}}$ be a cocycle and consider

$$\begin{aligned} (\pi_{\mathbb{C}})_*([V_{\mathbb{C}}]) \cdot B_{\mathbb{R}} \cdot Z &= V_{\mathbb{C}} \cdot (\pi_{\mathbb{C}})^{-1}(B_{\mathbb{R}} \cap Z) = (V_{\mathbb{C}} \cdot E_{\mathbb{R}}) \cap \pi^{-1}(Z) \\ &= (V_{\mathbb{R}} \cdot V_{\mathbb{R}}) \cap \pi^{-1}(Z) = \pi_*(V_{\mathbb{R}}) \cdot \pi_*(V_{\mathbb{R}}) \cap Z. \end{aligned}$$

Since this holds for any cycle, we have that

$$(\pi_{\mathbb{C}})_*([V_{\mathbb{C}}]) \cdot B_{\mathbb{R}} = \pi_*(V_{\mathbb{R}}) \cdot \pi_*(V_{\mathbb{R}}).$$

Now the general case follows from the fact that a general real holomorphic map f is the composition of an embedding $\tilde{f} : V \rightarrow V \times M$ given by $x \rightarrow (x, f(x))$ and the restriction of the projection $V \times M \rightarrow M$. \square

Remark 2.12. After proving the above result, we realized that it can be traced back to [1, cf. Theorem A, (b), p. 311], where this was proved mod 2, and is stated there as the identification

$$\tilde{H}_{\mathbb{C}\text{-alg}}^{2k}(M_{\mathbb{R}}) = \{\alpha^2 \mid \alpha \in H_{\text{alg}}^k(M_{\mathbb{R}})\},$$

where $\bar{H}_{\mathbb{C}\text{-alg}}^{2k}(M_{\mathbb{R}})$ is the set of pull-backs in $M_{\mathbb{R}}$ of the Poincaré dual classes in $M_{\mathbb{C}}$ of real algebraic submanifolds,⁶ and $H_{\text{alg}}^k(M_{\mathbb{R}})$ is the set of classes Poincaré dual to a real algebraic set. The proof in [1] is more algebraic, but proves also that any “square of an algebraic class” is induced by a complex class mod 2. However, we really need the integral coefficient case in order to get our estimates on the Pontriagin classes (mod 2 estimates would be useless here).

One should always be careful that $[V_{\mathbb{R}}] \cdot [V_{\mathbb{R}}]$ is only a square if $V_{\mathbb{R}}$ is orientable, as we see from the following example:

Example 2.13. (1) Let us consider the inclusion $\mathbb{R}P^n \rightarrow \mathbb{C}P^n$, and the pull-back of the generator $u \in H^2(\mathbb{C}P^n)$ to $H^2(\mathbb{R}P^n)$. Remember that $H^q(\mathbb{R}P^n) = \mathbb{Z}/2\mathbb{Z}$ for q even different from 0, n and is equal to \mathbb{Z} for $q = 0$ and for $q = n$ if n is even. The pull-back of u is then equal to the generator of $H^2(\mathbb{R}P^n)$. This generator can be identified with the class $a^2 \in H^2(\mathbb{R}P^n, \mathbb{Z}/2\mathbb{Z})$, where a is the generator of $H^1(\mathbb{R}P^n, \mathbb{Z}/2\mathbb{Z})$. Even though this class is a square in $H^2(\mathbb{R}P^n, \mathbb{Z}/2\mathbb{Z})$, it is not a square in $H^2(\mathbb{R}P^n, \mathbb{Z})$.

(2) Let Q be the manifold of degree 2^{n-1} ,

$$Q_{\mathbb{C}} = \{(z_0, \dots, z_{2n-1}) \mid z_0^2 - z_{2j}^2 - z_{2j+1}^2 = 0, j = 1, \dots, n-1\},$$

in $\mathbb{C}P^{2n}$. Then $Q_{\mathbb{R}} = T^{n-1}$. The pull-back of the hyperplane class, dual to $\mathbb{C}P^{2n-1}$, is given by the square of $T^{n-1} \cap \mathbb{R}P^{2n-1} = T^{n-2} \cup T^{n-2}$.

For $I = (a_1, \dots, a_q)$, let

$$S_I(k) = \{L \mid \dim(L \cap k^{n-k+i+a_i}) \geq i\} \subset G_q(k^n).$$

This is the Schubert cycle associated with the multi-index I . If $|I| = \sum_{i=1}^q a_i$, the cycle has dimension $|I|$. If 1_r denotes a sequence of r ones, $S_{1_r}(\mathbb{C})$ is Poincaré dual to the r th Chern class. More generally it is important to notice that the S_I are submanifolds with codimension 2 singularities (cf. [23]), so that according to Remark 2.2 the previous results apply.

Remark 2.14. The cycle $\tilde{S}_{1_r}(\mathbb{R})$, lift of $S_{1_r}(\mathbb{R})$ to the Grassmannian of oriented k -subspaces, $\tilde{G}_k(\mathbb{R}^n)$, is orientable in $\tilde{G}_k(\mathbb{R}^n)$ provided r and k are both even [23]. Note that $\tilde{G}_k(\mathbb{R}^n)$ being simply connected is orientable, and orientability and coorientability coincide in this case.

Note also that the $\tilde{S}_I(\mathbb{R})$ are invariant by the involution τ of $\tilde{G}_k(\mathbb{R}^n)$ that sends a space to the same one with opposite orientation. If this involution is orientation preserving, then $G_k(\mathbb{R}^n)$ will be orientable, and this happens precisely for k even. However, the involution changes the orientation of $S_I(\mathbb{R})$, except when I corresponds to the Euler class. So in the sequel, squares are never real squares, and in particular, one should not conclude that our Pontriagin classes are squares (see Example 3.13)!

Now let S be a positive linear combination of Schubert cycles, c_S the Poincaré dual class of $S(\mathbb{C})$ and o_S either the trivial bundle or the unique non-trivial local coefficients. We denote by j the inclusion $j : G_k(\mathbb{R}) \rightarrow G_k(\mathbb{C})$. Then $p_S = j^*(c_S)$. Note that if $c_S = \sum_{\alpha \in \{1,n\}^r} a_{\alpha} c_1^{\alpha_1} \cdots c_r^{\alpha_r}$ (the a_{α} need not be all non-negative), then⁷ $p_S = \sum_{\alpha \in \{1,n\}^r} a_{\alpha} p_2^{\alpha_1} \cdots p_{2r}^{\alpha_r}$. We denote by o the orientation bundle of $G_k(\mathbb{R}^n)$.

⁶ This should not be confused with $H_{\mathbb{C}\text{-alg}}^{2k}(M_{\mathbb{R}})$ obtained by pulling back the classes dual to any complex submanifold of $M_{\mathbb{C}}$.

⁷ Remember that only the p_{4k} are non-zero.

Corollary 2.15. *Let $S(\mathbb{C})$ be a positive linear combination of Schubert cycles in $G_q(\mathbb{C}^n)$, and $S(\mathbb{R})$ the analogous cycle in $G_q(\mathbb{R}^n)$. Consider the inclusion $i : G_q(\mathbb{R}^n) \rightarrow G_q(\mathbb{C}^n)$ and $\gamma_S \in H^{2\text{codim}(S)}(G_q(\mathbb{C}^n), \mathbb{Z})$ and let $\rho_S \in H^{\text{codim}(S)}(G_q(\mathbb{R}^n), o)$ be the cohomology classes Poincaré dual to $S(\mathbb{C})$ and $S(\mathbb{R})$. Then*

$$j^*(\gamma_S) = \rho_S^2.$$

In other words, the cohomology class Poincaré dual to $S(\mathbb{C})$ in $G_k(\mathbb{C}^n)$ induces on $G_k(\mathbb{R}^n)$ a cohomology class Poincaré dual of $S(\mathbb{R}) \cdot S(\mathbb{R})$.

Proof. It follows immediately from Proposition 2.8. Note that we assume here that $S(\mathbb{R})$ is coorientable, and according to Pontriagin, this is the case exactly when $S(\mathbb{R})$ represents a non-zero Pontriagin class. Had we taken $\mathbb{Z}/2\mathbb{Z}$ coefficients, we could also take the pull-backs of the odd Chern classes c_{2k+1} . \square

Now let $E \rightarrow M$ be a “real” holomorphic bundle, that is, there is a real bundle $E_{\mathbb{R}}$ over $M_{\mathbb{R}}$ such that

$$E|_{M_{\mathbb{R}}} = (E_{\mathbb{R}} \otimes \mathbb{C})$$

and the anti-holomorphic involution τ lifts to $E_{\mathbb{C}}$ in such a way that $E_{\mathbb{R}}$ is the set of fixed points. In a more algebraic setting, E comes from an algebraic locally free sheaf on real algebraic variety X . Assume now that $E_{\mathbb{C}} \rightarrow M_{\mathbb{C}}$ is generated by sections, that is there are sections s_1, \dots, s_m such that for each point $x \in M_{\mathbb{C}}$, $s_1(z), \dots, s_m(z)$ generate E_z . A holomorphic section is said to be real if $s(\tau(z)) = \bar{s}(z)$. Clearly, if E is generated by sections, it is generated by real holomorphic sections. Indeed, the sections $\Re(s_j) = \frac{s_j(z) + \bar{s}_j(\tau(z))}{2}$, $\Im(s_1) = \frac{s_1(z) - \bar{s}_1(\tau(z))}{2i}$ are real holomorphic and generate $E_{\mathbb{C}}$.

Now the real holomorphic sections induce a real holomorphic map $\tau_{\mathbb{C}} : M_{\mathbb{C}} \rightarrow G_k(\mathbb{C}^m)$, such that its restriction $\tau_{\mathbb{R}}$ to $M_{\mathbb{R}}$ has its image in $G_k(\mathbb{R}^m)$. Moreover, $(\tau_{\mathbb{C}})^*(U_{\mathbb{C}}) = E_{\mathbb{C}}$, $(\tau_{\mathbb{R}})^*(U_{\mathbb{R}}) = E_{\mathbb{R}}$.

This last remark together with Lemma 2.3 implies

Theorem 2.16. *Let $E_{\mathbb{C}}$ be a bundle generated by its sections over $M_{\mathbb{C}}$ and assume $E_{\mathbb{C}}$ to be real. Let $J = (j_1, \dots, j_q)$ be such that $4 \sum_{i=1}^q j_i = \dim M$. Then we have the inequality*

$$\left| \left\langle \prod_{i=1}^q p_{4j_i}(E_{\mathbb{R}}), [M_{\mathbb{R}}] \right\rangle \right| \leq \left\langle \prod_{i=1}^q c_{2j_i}^2(E_{\mathbb{C}}), [M_{\mathbb{C}}] \right\rangle$$

that we write for short as

$$|\langle p_J(E_{\mathbb{R}}), [M_{\mathbb{R}}] \rangle| \leq \langle c_J(E_{\mathbb{C}})^2, [M_{\mathbb{C}}] \rangle.$$

Proof. With i denoting the inclusion $i : M_{\mathbb{R}} \rightarrow M_{\mathbb{C}}$ we have $\tau_{\mathbb{C}} \circ i = j \circ \tau_{\mathbb{R}}$. Then if S is a cycle as in the previous corollary, we have

$$\begin{aligned} \langle p_S(E_{\mathbb{R}}), M_{\mathbb{R}} \rangle &= \langle \tau_{\mathbb{R}}^* j^*(PD(S(\mathbb{C}))), M_{\mathbb{R}} \rangle \\ &= \langle \tau_{\mathbb{R}}^*(PD(S(\mathbb{R}) \cdot PD(S(\mathbb{R}))), M_{\mathbb{R}} \rangle = \langle \tau_{\mathbb{R}}^*(PD(S(\mathbb{R})) \cdot \tau_{\mathbb{R}}^*(PD(S(\mathbb{R}))), M_{\mathbb{R}} \rangle. \end{aligned}$$

According to Proposition 2.5, we have the inequality

$$\begin{aligned} &|\langle \tau_{\mathbb{R}}^*(PD(S(\mathbb{R})) \cdot \tau_{\mathbb{R}}^*(PD(S(\mathbb{R}))), M_{\mathbb{R}} \rangle| \\ &\leq \langle \tau_{\mathbb{C}}^*(PD(S(\mathbb{C})) \cdot \tau_{\mathbb{C}}^*(PD(S(\mathbb{C}))), M_{\mathbb{C}} \rangle = \langle c_S^2, M_{\mathbb{C}} \rangle. \quad \square \end{aligned}$$

Remark 2.17. (a) Of course the same statement holds for S representing an effective ample cycle and, as a consequence of Giambelli's formula, this is the case for the positive combination of Schur polynomials⁸

$$P = \det(c_{a_i - i + j}) \quad \text{where } r \geq a_1 \geq a_2 \geq \dots \geq a_r \geq 0.$$

For example, we get that for a bundle generated by its sections over a surface,

$$|\langle p_1(E_{\mathbb{R}})^2 - p_2(E_{\mathbb{R}}), M_{\mathbb{R}} \rangle| \leq \langle c_1(E_{\mathbb{C}})^2 - c_2(E_{\mathbb{C}}), M_{\mathbb{C}} \rangle.$$

(b) Of course the same holds, with obvious changes, if $E_{\mathbb{C}}^*$ is generated by sections, since then $c_j(E_{\mathbb{C}}^*) = (-1)^j c_j(E_{\mathbb{C}})$, and $p_j(E_{\mathbb{R}}^*) = p_j(E_{\mathbb{R}})$.

Corollary 2.18 (Kharlamov [15]). *Let $M_{\mathbb{R}}^4$ be a real algebraic manifold, and assume that $T^*M_{\mathbb{C}}^4$ is generated by sections. If $\sigma(M_{\mathbb{R}}^4)$ is the signature of the four-dimensional manifold $M_{\mathbb{R}}^4$, then we have*

$$|3\sigma(M_{\mathbb{R}}^4)| \leq c_2^2(M_{\mathbb{C}}).$$

Proof. This is obvious once we notice that

$$\sigma(M_{\mathbb{R}}) = \frac{1}{3} \langle p_1(M_{\mathbb{R}}^4), M_{\mathbb{R}}^4 \rangle. \quad \square$$

A similar proof yields a result for Pontriagin classes, not only Pontriagin numbers, as follows:

Proposition 2.19. *With the assumptions of the previous theorem, let $\mu_{\mathbb{R}}$ and $\mu_{\mathbb{C}}$ be cohomology classes Poincaré dual to $Z_{\mathbb{R}}$ and $Z_{\mathbb{C}}$, where $Z_{\mathbb{C}}$ is a singular cycle in $M_{\mathbb{C}}$ with codimension two singularities. Then we have*

$$|\langle p_J(E_{\mathbb{R}}) \cup \mu_{\mathbb{R}}, [M_{\mathbb{R}}] \rangle| \leq \langle c_J(E_{\mathbb{C}})^2 \cup \mu_{\mathbb{C}}, [M_{\mathbb{C}}] \rangle.$$

Corollary 2.20. *Let E be a bundle of rank r , A a real positive line bundle and p such that $E \otimes A^p$ is globally generated. Then we have, as a consequence of the above theorem,*

$$c_k(E \otimes A^p) = \sum_{j=0}^k p^{k-j} c_1(A)^{k-j} \cdot c_j(E).$$

Taking p even, we have that $A|_{M_{\mathbb{R}}}$ is trivial; hence $p_{4r}(E_{\mathbb{R}} \otimes A_{\mathbb{R}}) = p_{4r}(E_{\mathbb{R}})$. Thus, defining $\Gamma_r(E; t) = \sum_{j=0}^r t^{r-j} \cdot c_j(E)$, we have

$$|\langle p_J(E_{\mathbb{R}}), [M_{\mathbb{R}}] \rangle| \leq \langle \Gamma_J^2(E_{\mathbb{C}}; pc_1(A)), [M_{\mathbb{C}}] \rangle.$$

⁸ And it follows from [13] that there are no other such classes. Note that [13] is also a basic ingredient of [9].

3. Bounds

Let X be an $n \geq 1$ -dimensional connected smooth subvariety of $\mathbb{C}P^m$ of degree d . We denote by \mathcal{L}_X the line bundle $\omega_X \otimes \mathcal{O}(n+2)$, where $\omega_X = \bigwedge^n T^*X$ is the canonical bundle. Recall that \mathcal{L}_X is ample.⁹ We denote by L_X , K_X and h the (first) Chern classes of \mathcal{L}_X , ω_X and $\mathcal{O}(1)$, respectively.

Proposition 3.1. *One has the inequalities $1 \leq h^{n-i} L_X^i \leq d^{i+1}$ for $i = 0, \dots, n$.*

Proof. The left-hand side inequality follows from the ampleness of both $\mathcal{O}(1)$ and \mathcal{L}_X .

Let us prove the other inequality by induction on n . For $i = 0$, this inequality reduces to the equality $h^n = d = \deg(X)$.

Assume that $n > 1$. Let H be a smooth (connected) hyperplane section of X . It is an $n-1$ -dimensional subvariety of $\mathbb{C}P^{m-1}$ of degree d . By adjunction, one has

$$\omega_H = \omega_X \otimes \mathcal{O}(1)|_H,$$

and therefore

$$(\mathcal{L}_X)|_H = \mathcal{L}_H. \quad (3.1)$$

First, one has the equalities (projection formula)

$$h^{n-i} L_X^i = h^{n-i-1} L_{X|H}^i = h^{n-1-i} L_H^i$$

for all $i < n$.

Because both $\mathcal{O}(1)$ and \mathcal{L}_X are ample, one has $h L_X^{n-1} > 0$, and therefore there exists some rational $\alpha \in \mathbb{Q}$ such that

$$(L_X + \alpha h) L_X^{n-1} = 0.$$

In other words, $L_X + \alpha h$ is a primitive class in $H^2(X, \mathbb{Q})$. By the Hodge index theorem, one gets

$$(L_X + \alpha h)^2 L_X^{n-2} \leq 0.$$

Thus, the real quadratic form

$$t \mapsto (L_X + th)^2 L_X^{n-2}$$

has non-negative discriminant (because it is negative for $t = 0$ and positive for large t). This gives the inequality

$$0 < L_X^n \leq \frac{(h L_X^{n-1})^2}{(h^2 L_X^{n-2})}.$$

Applying the above inequality to a smooth codimension k -plane section, we get

$$0 < h^k L_X^{n-k} \leq \frac{(h^{k+1} L_X^{n-k-1})^2}{(h^{k+2} L_X^{n-k-2})}.$$

⁹ One proves by induction that $K_X + (n+1)H$ is globally generated by sections by using the Kodaira vanishing theorem. More precise results can be found in [27].

Set now $l_k = \log(h^{n-k} L_X^k)$. Since $h^{n-k} L_X^k \geq 1$ we have $l_k \geq 0$, and the previous inequality implies that $l_k + l_{k-2} \leq 2l_{k-1}$, or else $l_k - l_{k-1}$ is decreasing.

We claim that $l_1 - l_0 \leq \log(d)$. Indeed, first one has certainly $l_0 = \log(h^n) = \log(d)$. Second, observe that l_1 is obtained for curves as follows. One could use Castelnuovo's inequalities, but let us just give these elementary bounds. Let C be a generic intersection of X and $(n-1)$ hyperplane. It is a genus g curve of degree d . By 3.1, one has

$$l_1 = \log(\deg(\mathcal{L}_C)) = \log(2g - 2 + 3d).$$

One just has to bound g in terms of d . A generic projection defines a birational morphism onto a plane degree d curve C . One deduces the inequality

$$1 - g = \chi(\mathcal{O}_X) \geq \chi(\mathcal{O}_C) = -d(d-3)/2$$

and therefore

$$l_1 \leq \log(d^2)$$

giving the inequality in this case.

Thus, $l_k \leq (k+1) \log(d)$ and this concludes our proof. \square

Remark 3.2. The referee suggested an alternative proof reducing to the hypersurface case by projection.

4. Chern classes

Using the splitting principle, we get the formula

$$c_i(E \otimes L) = \sum_j \binom{n-j}{i-j} c_1(L)^{i-j} c_j(E)$$

for E a (complex) vector bundle and L a line bundle on any variety. Applying this identity to $E = \Omega(2)$ and $L = \mathcal{O}(-2)$

$$c_i(\Omega) = \sum_j (-2)^{i-j} \binom{n-j}{i-j} h^{i-j} c_j(\Omega(2)). \quad (4.1)$$

Because $\Omega_{\mathbb{C}P^n}(2)$ is globally generated, its quotient $\Omega_X(2)$ is therefore nef. By Demailly et al. [9, Corollary 2.6], every Chern number is controlled by K_X and h or, what is the same, by L_X and h . More precisely for every multi-index $I = (i_1, \dots, i_r)$ of $r \leq n$ integers in $[1, \dots, n]$, one has

$$0 \leq c_I(\Omega(2)) h^{n-|I|} \leq c_1(\Omega_X(2))^{|I|} h^{n-|I|} = (L_X + (n-2)h)^{|I|} h^{n-|I|},$$

where $c_I = c_{i_1} \cdots c_{i_r}$. Let us assume $n > 1$ (the case $n = 1$ is left to the reader !). Using 3.1, we get the estimate

$$0 \leq c_I(\Omega(2)) h^{n-|I|} \leq d(d+n-2)^{|I|}. \quad (4.2)$$

To bound $c_I(\Omega)h^{n-|I|}$, let us denote the multi-index $(n, \dots, n) \in \mathbb{Z}^r$ by \underline{n} . Using 4.1, one gets

$$\begin{aligned} |c_I(\Omega)h^{n-|I|}| &\leq \sum_J 2^{|I|-|J|} \binom{\underline{n}-J}{I-J} h^{n-|J|} c_J(\Omega(2)) \\ &\leq \sum_J 2^{|I|-|J|} \binom{\underline{n}-J}{I-J} d(d+n-2)^{|J|} \text{ by (4.2)} \\ &= d(d+n-2)^{|I|} \sum_J \binom{\underline{n}-I+I-J}{\underline{n}-I} 2^{|I|-|J|} (d+n-2)^{|J|-|I|}. \end{aligned}$$

Applying the identity

$$\sum_i \binom{i+m}{m} t^i = \frac{1}{(1-t)^{m+1}} \quad \text{with } t^{-1} = (d+n-2)/2 \quad \text{and} \quad m = n - i_\alpha,$$

we obtain that the last sum is bounded by

$$\prod_\alpha \frac{1}{(1-t)^{n-i_\alpha+1}},$$

which in turn is less or equal than 2^{n^2} because $0 \leq t \leq 1/2$ provided $d+n \geq 6$. This yields the estimate (the cases where $d+n \leq 6$ can be dealt with by inspection)

$$|c_I(\Omega)h^{n-|I|}| \leq 2^{n^2} d(d+n-2)^{|I|}, \quad (4.3)$$

where $I = (i_1, \dots, i_r)$.

5. Proof of the main Theorem 1.1

Assume that X is a real smooth subvariety of degree d of some projective space $\mathbb{R}P^m$. Recall that this means that X is defined by polynomials with real coefficients and that the complex corresponding variety is smooth.

The (twisted) cotangent bundle $\Omega_X(2)$ is therefore a quotient of the twisted cotangent bundle $\Omega_P(2)$. Because the latter is globally generated (straightforward computation), so is $\Omega_X(2)$. By 2.20 and 4.3, we easily get the claimed inequality

$$|p_I(X(\mathbb{R}))| \leq 2^{n^2+3n} d(d+n-2)^n.$$

Once this is proved, the finiteness of the number of cobordism classes follows from the fact that the cobordism ring as the work of [30], completing the results of Thom, Rokhlin and Milnor, is determined by the Pontriagin numbers.

Remark 5.1. According to Remark (a), there are also inequalities for the Schur's polynomials. If

$$P(c_k) = \det(c_{a_i-i+j}) \text{ where } r \geq a_1 \geq a_2 \geq \dots \geq a_r \geq 0,$$

we have at least that for any Schur's polynomial P there is a universal function $F_P(d, n)$ such that

$$|\langle P(p_{2k}), X(\mathbb{R}) \rangle| \leq F_P(d, n).$$

6. Betti numbers

Let us explain how to obtain bounds for the Betti numbers of X using 4.3. It is important to note that the results of this section only depend on the complex degree of $M_{\mathbb{C}}$, that is the lowest degree of an ample (not necessarily real) line bundle. In fact this section is about bounding Betti numbers of complex projective manifolds, as functions of their degree, and the application to real manifolds is a consequence of the Smith–Thom inequality.

If $n = 1$, then we have

$$h^{1,0} = g(X) \leq \frac{(d-1)(d-2)}{2},$$

which gives

$$b(X) \leq 2 + (d-1)(d-2).$$

Let us assume that $n > 1$. Let H be a smooth hyperplane section as above.

By the Lefschetz Hyperplane theorem, we have the relations

$$b_i(X) = b_i(H) \quad \text{if } i < n-1 \quad \text{and} \quad b_{n-1}(X) \leq b_{n-1}(H).$$

By Poincaré duality, we have

$$b_i(X) = b_{2n-i}(X).$$

Therefore, one gets

$$b_i(X) \leq b_{i-2}(H) \quad \text{if } i > n.$$

It remains to control the middle term, $b_n(X)$. But the holomorphic Gauss–Bonnet formula (due to Chern) says

$$\chi(X) = c_n(T_X).$$

Notice that this formula follows easily from the Riemann–Roch theorem and from the well-known formula

$$c_n(T_X) = \text{td}(T_X) \sum_x (-1)^p \text{ch} \left(\bigwedge^p (\Omega_X) \right)$$

[12, Proposition 5.3]). We therefore get¹⁰

$$\begin{aligned}
 b_n(X) &\leq |\chi(X) \sum_{i \neq n} (-1)^i b_i(X)| \\
 &\leq |\chi(X) + \sum_{i \neq n, n-1, n+1} (-1)^i b_i(X) + (-1)^{n-1} (b_{n-1}(X) + b_{n+1}(X))| \\
 &= |\chi(X) - \chi(H) + (-1)^{n-2} b_{n-1}(H) + (-1)^{n-1} (b_{n-1}(X) + b_{n+1}(X))| \\
 &\leq |c_n(TX)| + |c_{n-1}(TH)| + |b_{n-1}(H) - 2b_{n-1}(X)| \\
 &\leq 2 \cdot 2^{n^2} d(d+n-2)^n + b_{n-1}(H)
 \end{aligned}$$

thanks to (4.3) and the fact that

$$-b_{n-1}(H) \leq b_{n-1}(H) - 2b_{n-1}(X) \leq b_{n-1}(H)$$

If $b(X) = \sum b_i(X)$ is the total Betti number, then one gets

$$b(X) \leq 3b(H) + 2 \cdot 2^{n^2} d(d+n-2)^n$$

and finally

replacing $3b(H)$ by $4b(H)$

$$b(X) \leq 4^n (d^2 + 1) + 2 \sum_{k=2}^n 4^{n-k} \cdot 2^{k^2} d(d+k-2)^k$$

To get a bound without summation, one can for instance bound $4^{n-k} \cdot 2^{k^2}$ by $2^{n+2+(k+1)(n-2)}$ and $(d+k-2)^k$ by $2^k d^k (k-2)^k$ for $k > 2$.

Then

$$\log_2 \left(\frac{4^{n-k} \cdot 2^{k^2} d(d+k-2)^k}{d^{k+1}} \right) \leq 2(n-k) + k^2 + k \log_2(k-2) \leq 2n + k(2n-1)$$

we get

$$2 \sum_{k=2}^n 4^{n-k} \cdot 2^{k^2} d(d+k-2)^k \leq 2d2^{2n} \sum_{k=3}^n (d2^{2n-1})^k$$

and checking by hand that the first three terms are greater than $4^n (d^2 + 1) + 24^{n-2} 2^4 d d^2$, we see that

$$b(X) \leq 2d2^{2n} \sum_{k=0}^n (d2^{2n-1})^k = d2^{2n+1} \left(\frac{d^{n+1} 2^{(2n-1)(n+1)} - 1}{d2^{2n-1} - 1} \right)$$

using the inequality

$$d2^{2n-1} - 1 \geq d2^{2n-2}$$

¹⁰ Notice that the Euler characteristic is independent from the coefficient field.

we get

$$b(X) \leq d 2^{2n+1} \frac{d^{n+1} 2^{(2n-1)(n+1)}}{d^{2n-2}} = 2^{2n^2+n+3} d^{n+1} \quad (6.1)$$

(of course it is easy to get a better bound¹¹, but with the same growth in d at fixed n).

Using the Smith–Thom’s inequality, one gets

Proposition 6.1. *Let X be a real subvariety of the projective space. Assume that $X_{\mathbb{C}}$ is smooth, connected to degree d and dimension n . Then, one has $b(X(\mathbb{R})) \leq 2^{n^2+n+3} d^{n+1}$.*

Notice that this bound does not depend on the real structure. Also the bound on mod 2 Betti numbers of course implies the same on rational Betti numbers.

Remark 6.2. It should be interesting to have some control on the fundamental group of the (non-connected) real part. For instance, given a finite covering of $X(\mathbb{R})$, can one lift this covering to a finite ramified covering of $X_{\mathbb{C}}$ ramified over an hypersurface of controlled degree (without a real point of course)?

Acknowledgements

We are very grateful to Slava Kharlamov, for the many suggestions, his constant encouragement, as well as for copious information on the history of the Real Algebraic Geometry. We thank Arnaud Beauville and Jean-Pierre Demailly for useful discussions and advice.

Appendix A. Some simple finiteness results

Let X be an n -dimensional complex algebraic variety. We denote by \bar{X} the conjugate variety obtained by the base change by the conjugation $\mathbb{C} \rightarrow \mathbb{C}$. Explicitly, if X is (locally) defined by polynomial equations $\sum a_{i,j} x^i = 0$, then the variety \bar{X} is defined by the conjugate equations $\sum \bar{a}_{i,j} x^i = 0$. Of course, one has $\bar{\bar{X}} = X$. To give a real structure on X remains to give a complex morphism $t : X \rightarrow \bar{X}$ such that $\bar{t} \circ t = \text{Id}_X$ (we will simply say that t is a skew involution of X). The real points of this real complex structure will be denoted by $X_t(\mathbb{R})$ (the set of fixed points). If t_1 and t_2 are two skew involutions, the composite $t_1 \circ \bar{t}_2$ belongs to $\text{Aut}(X)$. We therefore get the following well-known result.

Proposition A.1. *A complex variety of general types has a finite number of real structures.*

Proof. Observe that $\text{Aut}(X)$ is finite in this case [16]. \square

¹¹ In fact Kharlamov and Zak improved this bound to $b(X) \leq d^{n+1}$ (see: V. Kharlamov, Overview of topological properties of real algebraic surfaces AG/0502127).

A.1. Final remarks

The set of morphisms $\text{Hom}(X, \bar{X})$ is parameterized by the complex points of a countable union of quasiprojective varieties, which is a locally closed subvariety of the Hilbert scheme of $X \times \bar{X}$ (look at the graph), and therefore the set of skew morphisms is the same property. Recall (Chow) that the subvariety of the Hilbert scheme of a smooth projective (polarized) variety parameterizing the reduced subvarieties of bounded degree is quasiprojective. If L is an ample line bundle on X , it defines a conjugate line bundle \bar{L} on \bar{X} , and therefore $X \times \bar{X}$ is polarized by $L \boxtimes \bar{L}$. The degree of the graph of a morphism $t \in \text{Hom}(X, \bar{X})$ is $(c_1(L) + t^*c_1(\bar{L}))^n$. In particular, if $t^*\bar{L}$ and L are numerically equivalent, this degree is bounded by $2^n \deg_L(X)$. This is certainly the case if for instance $H^{1,1}(X) = \mathbb{C}$. Another good situation is when X is Fano. Indeed, one can take $L = \omega_X^{-1}$ (observe that $\bar{\omega}_X = \omega_{\bar{X}}$).

Corollary A.2. *The real structures of a given complex Fano variety have a finite number of deformation classes.*

Remark A.3. If the automorphism group of X is linear algebraic, its action on the Picard group factors through the finite group of the connected components. One deduces (averaging a given ample bundle) that the number of deformation classes is finite in this case. A more general result using [4] can be found in [7, Section D.1.10].

In particular, the number of diffeomorphisms types of $X_t(\mathbb{R})$ when t runs over all real structures is then finite. This is for instance the case for toric varieties (due to Demazure). This last observation was first made in [8]. The same averaging process can more generally be achieved if for instance the closure of the ample cone is (rational) polyhedral (which is also the case for toric varieties), giving a more elementary proof of the former finiteness result (use the fact that the one-dimensional edges are permuted by the automorphisms and that an automorphism fixing the edges is the identity because it is an integral dilatation on each edge on the one hand and, on the other hand, it is volume preserving, its determinant being equal to ± 1).

References

- [1] S. Akbulut, H. King, Transcendental submanifolds of \mathbb{R}^n , *Comment. Math. Helv.* 68 (1993) 308–318.
- [2] A. Borel, *Seminar on Transformation Groups*, (with contributions by G. Bredon, E.E. Floyd, D. Montgomery, R. Palais). *Annals of Math Studies*, vol. 46, Princeton University Press, Princeton, NJ, 1960.
- [3] A. Borel, A. Haefliger, La classe d'homologie fondamentale d'un espace analytique, *Bull. Soc. Math. France* 89 (1964) 461–513.
- [4] A. Borel, J.P. Serre, Théorèmes de finitude en cohomologie galoisienne, *Comment. Math. Helv.* 39 (1964) 111–164.
- [5] A. Degtyarev, I. Itenberg, V. Kharlamov, *Real Enriques Surfaces*, *Lecture Notes in Mathematics*, vol. 1746, Springer, Berlin, 2000.
- [6] A. Degtyarev, I. Itenberg, V. Kharlamov, Topological properties of real algebraic varieties: du côté de chez Rokhlin, *Uspekhi Matematicheskikh Nauk*, 2, 2000.
- [7] A. Degtyarev, I. Itenberg, V. Kharlamov, Finiteness and quasi-simplicity for symmetric $k3$ -surfaces, *Duke Math. J.* 122 (2004) 1–49.
- [8] C. Delaunay, Structures réelles sur les variétés toriques compactes. Ph.D. Thesis, Université Louis Pasteur, Strasbourg, 2004.
- [9] J.-P. Demailly, T. Peternell, M. Schneider, Compact complex manifolds with numerically effective tangent bundles, *J. Algebraic Geom.* 3 (1994) 295–345.

- [10] Y. Eliashberg, A. Givental, H. Hofer, Introduction to symplectic field theory, *Geom. Funct. Anal.* Part II (2000) 560–573 (Special volume).
- [11] E.E. Floyd, On periodic maps and the Euler characteristics of associated spaces, *Trans. Amer. Math. Soc.* 72 (1952) 138–147.
- [12] W. Fulton, S. Lang, Riemann–Roch algebra, *Grundlehren der Mathematischen Wissenschaften*, vol. 277, Springer, New York, Heidelberg, Berlin, 1985.
- [13] W. Fulton, R. Lazarsfeld, Positive polynomials for ample vector bundles, *Ann. of Math.* (2) 118 (1983) 35–60.
- [14] S. Kharlamov, Variétés de Fano réelles [d’après C. Viterbo], mars 2000, exposé 872, Séminaire Bourbaki, Astérisque 276 (2002).
- [15] S. Kharlamov, private communication, Berkeley, Mars, 2004.
- [16] S. Kobayashi, On the automorphism group of a certain class of algebraic manifolds, *Tôhoku Math. J.* 11 (1959) 184–190.
- [17] J. Kollár, Real algebraic threefolds ii. minimal model program, *J. Amer. Math. Soc.* 12 (1999) 33–83.
- [18] J. Kollár, Which are the simplest algebraic varieties?, *Bull. Amer. Math. Soc. (New Series)* 38 (2001) 409–433.
- [19] J. Milnor, On the Betti numbers of real varieties, *Proc. Amer. Math. Soc.* 15 (1964) 275–280.
- [20] D. Mumford, Varieties defined by quadratic equations. in: *Questions on Algebraic Varieties (C.I.M.E., II ciclo, Varenna 1969)*, Edizione Cremonese, Roma, 1969, pp. 29–100.
- [21] O.A. Oleinik, Estimates of the Betti numbers of real algebraic hypersurfaces, *Mat. Sbornik (N.S.)* 28 (70) (1951) 635–640.
- [22] I.B. Petrowski, O.A. Oleinik, On the topology of real algebraic surfaces, *Izv. Akad. nauk. SSSR* 13 (1949) 389–402.
- [23] L.S. Pontryagin, Characteristic cycles for differentiable manifolds, *Mat. Sbornik (N.S.)* 21 (2) (1947) 233–284 (reprinted in L.S. Pontryagin Selected Research papers, pp. 375–433).
- [24] M. Richardson, P.A. Smith, Periodic transformations of complexes, *Ann. of Math.* 39 (1938) 611–633.
- [25] R. Silhol, *Real Algebraic Surfaces*, Springer, Berlin, 1989.
- [26] P.A. Smith, The topology of transformation groups, *Bull. Amer. Math. Soc.* 44 (1938) 497–514.
- [27] A.J. Sommese, A. Van de Ven, On the adjunction mapping, *Math. Ann.* 278 (1–4) (1987) 593–603.
- [28] R. Thom, Quelques propriétés des variétés bords, in: *Colloque de Topologie de Strasbourg*, 1951.
- [29] R. Thom, Sur l’homologie des variétés algébriques réelles, in: S.S. Cairns (Ed.), *Differential and Combinatorial Topology*, Princeton University Press, Princeton, NJ, 1965, pp. 255–265.
- [30] C.T.C. Wall, Determination of the cobordism ring, *Ann. of Math.* 72 (1960) 292–311.